UNIVERSIDAD SAN IGNACIO DE LOYOLA

Optimización

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Contenido

Optimización Estática

❖ Problema general sin restricciones

$$\underset{x_i}{Max} \ f(x_1, x_2, ..., x_n) \ \rightarrow \ i = 1,2,...,n$$

Condiciones de Primer Orden:

$$\frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_i} = 0 \ \rightarrow \ i = 1,2,...,n$$

Condiciones de Segundo Orden:

$$H\left[f(x_1, x_2, ..., x_n)\right]_{x=x^*}$$

Optimización Estática

❖ Problema general con restricciones de igualdad

$$\underset{x_i}{Max} \ f(x_1, x_2, ..., x_n) \ \rightarrow \ i = 1,2,...,n$$

$$s.a.: \ g_j(x_1, x_2, ..., x_n) = 0 \ \rightarrow \ j = 1,2,...,m$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^{m} \lambda_j g_j(x_i)$

$$\frac{\partial L}{\partial x_i} = \frac{\partial f(x_i)}{\partial x_i} + \sum_{j=1}^{m}\left[\lambda_j \frac{\partial g_j(x_i)}{\partial x_i}\right] = 0 \ \rightarrow \ i = 1,2,...,n$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(x_1, x_2, ..., x_n) = 0 \ \rightarrow \ j = 1,2,...,m$$

Condiciones de Segundo Orden:

$$H\left[f(x_1, x_2, ..., x_n)\right]_{x=x^*}$$

$$H\left[g(x_1, x_2, ..., x_m)\right]_{x=x^*}$$

Optimización Estática

❖ **Problema general con restricciones de desigualdad**

$$\underset{x_i}{Max} \; f(x_1, x_2, ..., x_n) \;\rightarrow\; i = 1,2,...,n$$

$$s.a.: \; g_j(x_1, x_2, ..., x_n) \geq 0 \;\rightarrow\; j = 1,2,...,m$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^{m} \lambda_j g_j(x_i)$

$$\frac{\partial L}{\partial x_i} = 0 \;\rightarrow\; i = 1,2,...,n$$

$$\lambda_j \geq 0 \qquad \frac{\partial L}{\partial \lambda_j} \geq 0 \qquad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0 \;\rightarrow\; j = 1,2,...,m$$

Condiciones de Segundo Orden:

$$H[f(x_1, x_2, ..., x_n)]_{x=x^*}$$

$$H[g(x_1, x_2, ..., x_m)]_{x=x^*}$$

Optimización Estática

❖ **Problema general con restricciones de desigualdad y no negatividad**

$$\underset{x_i}{Max} \; f(x_1, x_2, ..., x_n) \;\rightarrow\; i = 1,2,...,n$$

$$s.a.: \; g_j(x_1, x_2, ..., x_n) \geq 0 \;\rightarrow\; j = 1,2,...,m$$

$$x_i \geq 0 \;\rightarrow\; i = 1,2,...,n$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^{m} \lambda_j g_j(x_i)$

$$x_i \geq 0 \qquad \frac{\partial L}{\partial x_i} \leq 0 \qquad x_i \frac{\partial L}{\partial x_i} = 0 \;\rightarrow\; i = 1,2,...,n$$

$$\lambda_j \geq 0 \qquad \frac{\partial L}{\partial \lambda_j} \geq 0 \qquad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0 \;\rightarrow\; j = 1,2,...,m$$

Condiciones de Segundo Orden:

$$H[f(x_1, x_2, ..., x_n)]_{x=x^*}$$

$$H[g(x_1, x_2, ..., x_m)]_{x=x^*}$$

Contenido

Cálculo de variaciones

❖ **Problema general**

$$MaxV = \int_0^T f(t, y_t, \dot{y}_t)\partial t$$

$$s.a.: \quad y(0) = y_0$$
$$y(T) = y_T$$

Condiciones de Primer Orden:

$$f_y = \frac{\partial f_{\dot{y}}}{\partial t}$$

Condiciones de Segundo Orden:

$$H\left[f(t, y_t, \dot{y}_t)\right]_{(y^*, \dot{y}^*)}$$

Cálculo de variaciones

❖ **Ejemplo 1: Hallar la solución de**

$$Max\ V[y_t]=\int_0^1 (\dot{y}_t^2-2y_t\dot{y}_t+10ty_t)\partial t$$

$$s.a.:\quad y_0=1$$

$$y_1=2$$

Condiciones de Primer Orden:

$$f_y=\frac{\partial f_{\dot{y}}}{\partial t}$$

$$f_y=-2\dot{y}_t+10t$$

$$f_{\dot{y}}=2\dot{y}_t-2y_t$$

$$\frac{\partial f_{\dot{y}}}{\partial t}=2\ddot{y}_t-2\dot{y}_t$$

$$\ddot{y}_t=5t$$

$$y_t=\frac{5}{6}t^3+H_1t+H_2$$

Cálculo de variaciones

❖ **Ejemplo 1: Hallar la solución de**

$$Max\ V[y_t]=\int_0^1 (\dot{y}_t^2-2y_t\dot{y}_t+10ty_t)\partial t$$

$$s.a.:\quad y_0=1$$

$$y_1=2$$

Condiciones de Primer Orden:

$$y_t^*=\frac{5}{6}t^3+H_1t+H_2$$

$$y_t^*=\frac{5}{6}t^3+\frac{1}{6}t+1$$

Cálculo de variaciones

❖ **Condición de transversalidad**

$$\left[f_{\dot{y}}\right]_{t=T}\Delta y_T+\left[f-\dot{y}\ f_{\dot{y}}\right]_{t=T}\Delta T=0$$

Horizonte temporal fijo:

$$\left[f_{\dot{y}}\right]_{t=T}=0$$

Valor terminal fijo:

$$\left[f-\dot{y}\ f_{\dot{y}}\right]_{t=T}=0$$

Curva terminal:

$$\left[f_{\dot{y}}\right]_{t=T}g'(T)+\left[f-\dot{y}\ f_{\dot{y}}\right]_{t=T}=0$$

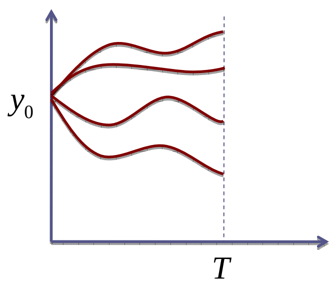
Cálculo de variaciones

❖ **Condición de transversalidad**

$$\left[f_{\dot{y}}\right]_{t=T}\Delta y_T+\left[f-\dot{y}\ f_{\dot{y}}\right]_{t=T}\Delta T=0$$

Horizonte terminal fijo:

$$\left[f_{\dot{y}}\right]_{t=T}=0$$



Cálculo de variaciones

- **Ejemplo 2: horizonte terminal fijo**

$$Max\ V[y_t] = \int_0^2 (t^2 + \dot{y}_t^2)\partial t$$

$$s.a.: \quad y_0 = 4$$

$$y_2 = libre$$

Condiciones de Primer Orden:

$$f_y = \frac{\partial f_{\dot{y}}}{\partial t}$$

$$f_y = 0$$

$$f_{\dot{y}} = 2\dot{y}_t$$

$$\frac{\partial f_{\dot{y}}}{\partial t} = 2\ddot{y}_t$$

$$\ddot{y}_t = 0$$

$$y_t = H_1 t + H_2$$

Cálculo de variaciones

- **Ejemplo 2: horizonte terminal fijo**

$$Max\ V[y_t] = \int_0^2 (t^2 + \dot{y}_t^2)\partial t$$

$$s.a.: \quad y_0 = 4$$

$$y_2 = libre$$

Condiciones de Primer Orden:

$$y_t = H_1 t + H_2$$

Utilizando la condición inicial

$$y_t = H_1 t + 4$$

Cálculo del segundo parámetro

$$\left[f_{\dot{y}} \right]_{t=2} = 0$$

$$f_{\dot{y}} = 2\dot{y}_t = 2H_1$$

$$[2H_1]_{t=2} = 0$$

$$y_t^* = 4$$

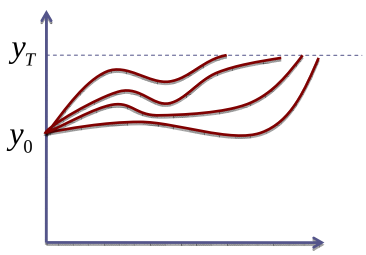
Cálculo de variaciones

❖ **Condición de transversalidad**

$$\left[f_{\dot{y}}\right]_{t=T}\Delta y_T+\left[f-\dot{y}\ f_{\dot{y}}\right]_{t=T}\Delta T=0$$

Valor terminal fijo:

$$\left[f-\dot{y}\ f_{\dot{y}}\right]_{t=T}=0$$



Cálculo de variaciones

❖ **Ejemplo 3: valor terminal fijo**

$$Max\ V[y_t]=\int_0^T(t^2+\dot{y}_t^2)\partial t$$

$$s.a.:\quad y_0=4$$

$$y_T=5$$

Condiciones de Primer Orden:

$$f_y=\frac{\partial f_{\dot{y}}}{\partial t}$$

$$f_y=0$$

$$f_{\dot{y}}=2\dot{y}_t$$

$$\frac{\partial f_{\dot{y}}}{\partial t}=2\ddot{y}_t$$

$$\ddot{y}_t=0$$

$$y_t=H_1t+H_2$$

Cálculo de variaciones

❖ **Ejemplo 3: valor terminal fijo**

$$Max\ V[y_t] = \int_0^T (t^2 + \dot{y}_t^2)\partial t$$

$$s.a.: \quad y_0 = 4$$

$$y_T = 5$$

Condiciones de Primer Orden: $y_t = H_1 t + H_2$

Utilizando la condición inicial $y_t = H_1 t + 4$

Cálculo del segundo parámetro $\left[f - \dot{y}\ f_{\dot{y}} \right]_{t=T} = 0$

$$f - \dot{y}\ f_{\dot{y}} = t^2 + \dot{y}_t^2 - \dot{y}_t(2\dot{y}_t) = t^2 - \dot{y}_t^2$$

$$\left[t^2 - H_1^2 \right]_{t=T} = 0$$

$$H_1 = T$$

$$y_T = H_1 T + 4 = H_1 H_1 + 4 = 5$$

$$y_t^* = t + 4$$

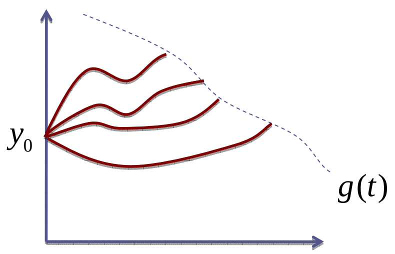
Cálculo de variaciones

❖ **Condición de transversalidad**

$$\left\lfloor f_{\dot{y}} \right\rfloor_{t=T} \Delta y_T + \left\lfloor f - \dot{y}\ f_{\dot{y}} \right\rfloor_{t=T} \Delta T = 0$$

Curva terminal fija:

$$\left\lfloor f_{\dot{y}} \right\rfloor_{t=T} g'(T) + \left\lfloor f - \dot{y}\ f_{\dot{y}} \right\rfloor_{t=T} = 0$$



Cálculo de variaciones

❖ **Ejemplo 4: curva terminal fija**

$$Max\ V[y_t] = \int_0^T (t^2 + \dot{y}_t^2)\partial t$$

$$s.a.: \quad y_0 = 4$$

$$y_T \ sobre\ la\ curva\ g(t) = 1 - \frac{t^2}{4}$$

Condiciones de Primer Orden:

$$f_y = \frac{\partial f_{\dot{y}}}{\partial t}$$

$$f_y = 0$$

$$f_{\dot{y}} = 2\dot{y}_t$$

$$\frac{\partial f_{\dot{y}}}{\partial t} = 2\ddot{y}_t$$

$$\ddot{y}_t = 0$$

$$y_t = H_1 t + H_2$$

Cálculo de variaciones

❖ **Ejemplo 4: curva terminal fija**

$$Max\ V[y_t] = \int_0^T (t^2 + \dot{y}_t^2)\partial t$$

$$s.a.: \quad y_0 = 4$$

$$y_T \ sobre\ la\ curva\ g(t) = 1 - \frac{t^2}{4}$$

Condiciones de Primer Orden:

$$y_t = H_1 t + H_2$$

Utilizando la condición inicial

$$y_t = H_1 t + 4$$

Cálculo del segundo parámetro

$$\left[f_{\dot{y}} \right]_{t=T} g'(T) + \left[f - \dot{y}\ f_{\dot{y}} \right]_{t=T} = 0$$

$$f_{\dot{y}} = 2\dot{y}_t = 2H_1$$

$$f - \dot{y}\ f_{\dot{y}} = t^2 + \dot{y}_t^2 - \dot{y}_t(2\dot{y}_t) = t^2 - \dot{y}_t^2 = t^2 - 2H_1^2$$

$$2H_1^2 + TH_1 - T^2 = 0 \qquad H_1 = (0.5T, -T)$$

Cálculo de variaciones

❖ **Ejemplo 4: curva terminal fija**

$$Max\ V[y_t] = \int_0^T (t^2 + \dot{y}_t^2)\partial t$$

$$s.a.:\quad y_0 = 4$$

$$y_T \text{ sobre la curva } g(t) = 1 - \frac{t^2}{4}$$

Condiciones de Primer Orden:

$$y_t = H_1 t + 4$$

$$H_1 = -T$$

$$y_t = -Tt + 4$$

$$g(T) = 1 - \frac{T^2}{4} \qquad y_T = -T^2 + 4$$

$$1 - \frac{T^2}{4} = -T^2 + 4$$

$$T = 2$$

$$y_t^* = -2t + 4$$

Cálculo de variaciones

❖ **Ejemplo 4: curva terminal fija**

$$Max\ V[y_t] = \int_0^T (t^2 + \dot{y}_t^2)\partial t$$

$$s.a.:\quad y_0 = 4$$

$$y_T \text{ sobre la curva } g(t) = 1 - \frac{t^2}{4}$$

Condiciones de Primer Orden:

$$y_t = H_1 t + 4$$

$$H_1 = 0.5T$$

$$y_t = 0.5Tt + 4$$

$$g(T) = 1 - \frac{T^2}{4} \qquad y_T = 0.5T^2 + 4$$

$$1 - \frac{T^2}{4} = 0.5T^2 + 4$$

$$T = \pm\sqrt{2}i$$

Cálculo de variaciones

❖ Condición de segundo orden

$$H\left[f(t, y_t, \dot{y}_t)\right]_{(y^*, \dot{y}^*)}$$

Método de menores principales

Método de raíces características

Cálculo de variaciones

❖ Ejemplo 5: Minimización de costos

- Una fábrica recibe el pedido de B unidades (no hay plazo determinado)
- Costos cuadráticos de producción y costo de almacenaje proporcional a la producción acumulada.
- Definimos q la producción en período t, y la producción acumulada.

$$C_1(q_t) = aq_t^2 \qquad\qquad C_2(y_t) = by_t$$

- Teniendo en cuenta

$$q_t = \dot{y}_t$$

- Costo total

$$C(y_t) = a\dot{y}_t^2 + by_t$$

Cálculo de variaciones

❖ **Ejemplo 5: Minimización de costos**

$$Max\ V[y_t] = -\int_0^T (a\dot{y}_t^2 + by_t)\partial t$$

$$s.a.: \quad y_0 = 0$$

$$y_T = B$$

Condiciones de Primer Orden:

$$f_y = \frac{\partial f_{\dot{y}}}{\partial t}$$

$$f_y = -b$$

$$f_{\dot{y}} = -2a\dot{y}_t$$

$$\frac{\partial f_{\dot{y}}}{\partial t} = -2a\ddot{y}_t$$

$$2a\ddot{y}_t = b$$

$$y_t^* = \frac{b}{4a}t^2 + H_1 t + H_2$$

Cálculo de variaciones

❖ **Ejemplo 5: Minimización de costos**

$$Max\ V[y_t] = -\int_0^T (a\dot{y}_t^2 + by_t)\partial t$$

$$s.a.: \quad y_0 = 0$$

$$y_T = B$$

Condiciones de Primer Orden:

$$y_t^* = \frac{b}{4a}t^2 + H_1 t + H_2$$

Utilizando la condición inicial

$$y_t^* = \frac{b}{4a}t^2 + H_1 t$$

Cálculo del segundo parámetro

$$\left[f - \dot{y}\ f_{\dot{y}} \right]_{t=T} = 0$$

$$f - \dot{y}\ f_{\dot{y}} = -a\dot{y}_t^2 - by_t + \dot{y}_t(2a\dot{y}_t) = -by_t + a\dot{y}_t^2$$

$$\left[-\frac{b^2}{4a}t^2 - H_1 bt + a\left(\frac{b}{2a}t + aH_1 \right)^2 \right]_{t=T} = 0$$

Cálculo de variaciones

- **Ejemplo 5: Minimización de costos**

$$Max\ V[y_t] = -\int_0^T (a\dot{y}_t^2 + by_t)\partial t$$
$$s.a.:\quad y_0 = 0$$
$$y_T = B$$

Condiciones de Primer Orden:

$$-\frac{b^2}{4a}T^2 - H_1 bT + \frac{b^2}{4a}T^2 + abTH_1 + a^3H_1^2 = 0$$

$$H_1(-bT + abT + a^3H_1) = 0$$

Senda Óptima:

$$y_t^* = \frac{b}{4a}t^2$$

$$T = 2\sqrt{\frac{aB}{b}}$$

Cálculo de variaciones

- **Ejemplo 5: Minimización de costos**

$$Max\ V[y_t] = -\int_0^T (a\dot{y}_t^2 + by_t)\partial t$$
$$s.a.:\quad y_0 = 0$$
$$y_T = B$$

Condiciones de Segundo Orden:

$$y_t^* = \frac{b}{4a}t^2 \qquad T = 2\sqrt{\frac{aB}{b}}$$

Matriz Hessiana:

$$H[y_t, \dot{y}_t] = \begin{bmatrix} 0 & 0 \\ 0 & -2a \end{bmatrix}_{t=T}$$

¿Métodos?

Cálculo de variaciones

- **Horizonte temporal infinito**

$$MaxV = \int_0^{\infty} f(t, y_t, \dot{y}_t)\partial t$$

$$s.a.: \quad y(0) = y_0$$

Condiciones de convergencia de la función objetivo:

1. Si la función V posee un valor finito hasta T, y luego toma el valor de cero, entonces la integral converge.

2. Si la función f es descontada a través del factor e^{-pt} (p>0) y tiene un valor menor o igual a M (finito), entonces la integral converge

Cálculo de variaciones

- **Problema general**

$$MaxV = \int_0^{\infty} f(t, y_t, \dot{y}_t)\partial t$$

$$s.a.: \quad y(0) = y_0$$

Condiciones de Primer Orden:

$$f_y = \frac{\partial f_{\dot{y}}}{\partial t}$$

Condiciones de Segundo Orden:

$$H\left[f(t, y_t, \dot{y}_t)\right]_{(y^*, \dot{y}^*)}$$

Cálculo de variaciones

❖ **Condición de transversalidad**

$$\lim_{T\to\infty}\left\{\left[f_{\dot{y}}\right]_{t=T}\Delta y_T+\left[f-\dot{y}\ f_{\dot{y}}\right]_{t=T}\Delta T\right\}=0$$

Debe cumplirse la condición:

$$\lim_{t\to\infty}\left[f_{\dot{y}}\right]=0$$

Valor terminal no especificado:

$$\lim_{t\to\infty}\left[f-\dot{y}\ f_{\dot{y}}\right]=0$$

Meta asintótica:

$$\lim_{t\to\infty} y_t = y_\infty$$

Contenido

Control Óptimo

❖ **Problema general**

$$MaxV = \int_0^T f(t, y_t, u_t)\partial t$$

$$s.a.: \quad \dot{y}_t = g(t, y_t, u_t)$$

$$y(0) = y_0$$

$$y(T) = y_T$$

Condiciones de Primer Orden:

$$H(t, y_t, u_t) = f(t, y_t, u_t) + \lambda_t g(t, y_t, u_t)$$

$$\underset{u_t}{Max}\, H(t, y_t, u_t)$$

$$\frac{\partial H}{\partial y_t} = -\dot{\lambda}_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{y}_t$$

$$\lambda_T = 0$$

Condiciones de Segundo Orden:

$$Hes[H(t, y_t, u_t)]_{(y^*, u^*)}$$

Control Óptimo

❖ **Ejemplo 1: Hallar la solución de**

$$Max\ V[y_t] = \int_0^1 \frac{u_t^{1-\theta}}{1-\theta}\partial t$$

$$s.a.: \quad \dot{y}_t = -y_t - u_t$$

$$y_0 = 2$$

$$y_1 = 1$$

Condiciones de Primer Orden:

$$H(t, y_t, u_t, \lambda_t) = \frac{u_t^{1-\theta}}{1-\theta} + \lambda_t(-y_t - u_t)$$

$$\frac{\partial H}{\partial u_t} = u_t^{-\theta} - \lambda_t = 0$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial y_t} = \lambda_t$$

$$\dot{y}_t = \frac{\partial H}{\partial \lambda_t} = -y_t - u_t$$

$$\lambda_t^* = H_3 e^t$$

$$u_t^* = \left(H_3 e^t\right)^{-\frac{1}{\theta}} = H_1 e^{-\frac{t}{\theta}}$$

$$y_t^* = H_2 e^{-t} + \frac{\theta}{\theta - 1} H_1 e^{-\frac{t}{\theta}}$$

Control Óptimo

❖ **Ejemplo 1: Hallar la solución de**

$$Max\ V[y_t] = \int_0^1 \frac{u_t^{1-\theta}}{1-\theta}\partial t$$

$$s.a.:\quad \dot{y}_t = -y_t - u_t$$

$$y_0 = 2$$

$$y_1 = 1$$

Condiciones de Primer Orden:

$$y_t^* = H_2 e^{-t} + \frac{\theta}{\theta-1} H_1 e^{-\frac{t}{\theta}}$$

$$y_0^* = H_2 + \frac{\theta}{\theta-1} H_1 = 2$$

$$y_1^* = H_2 e^{-1} + \frac{\theta}{\theta-1} H_1 e^{-\frac{1}{\theta}} = 1$$

$$H_1 = \frac{-(1-\theta)(1-2e^{-1})}{\theta\left(e^{-\frac{1}{\theta}} - e^{-1}\right)}$$

$$H_2 = \frac{2e^{-\frac{1}{\theta}} - 1}{e^{-\frac{1}{\theta}} - e^{-1}}$$

Control Óptimo

❖ **Condición de transversalidad**

$$[H]_{t=T}\Delta T - \lambda_T \Delta y_T = 0$$

Horizonte temporal fijo:

$$\lambda_T = 0$$

Valor terminal fijo:

$$[H]_{t=T} = 0$$

Curva terminal:

$$[H]_{t=T} - \lambda_T g'(T) = 0$$

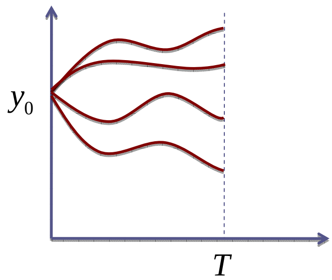
Control Óptimo

❖ **Condición de transversalidad**

$$[H]_{t=T}\,\Delta T - \lambda_T \Delta y_T = 0$$

Horizonte terminal fijo:

$$\lambda_T = 0$$



Control Óptimo

❖ **Ejemplo 2: horizonte terminal fijo**

$$\begin{aligned}
&Max\ V[y_t] = \int_0^2 (t^2 - u_t^2)\partial t \\
&s.a.: \quad \dot{y}_t = u_t \\
&\qquad\quad y_0 = 4 \\
&\qquad\quad y_2 = libre
\end{aligned}$$

Condiciones de Primer Orden:

$$H = t^2 - u_t^2 + \lambda_t u_t$$

$$\frac{\partial H}{\partial u_t} = -2u_t + \lambda_t = 0$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial y_t} = 0$$

$$\dot{y}_t = \frac{\partial H}{\partial \lambda_t} = u_t$$

$$\lambda_t^* = H_1$$

$$u_t^* = \frac{H_1}{2}$$

$$y_t^* = \frac{H_1}{2} t + H_2$$

Control Óptimo

❖ **Ejemplo 2: horizonte terminal fijo**

$$Max\ V[y_t] = \int_0^2 (t^2 - u_t^2)\partial t$$

$$s.a.: \quad \dot{y}_t = u_t$$

$$y_0 = 4$$

$$y_2 = libre$$

Condiciones de Primer Orden: $y_t^* = \frac{H_1}{2} t + H_2$

Utilizando la condición inicial $y_t^* = \frac{H_1}{2} t + 4$

Cálculo del segundo parámetro $\lambda_T = H_1 = 0$

$$y_t^* = 4$$

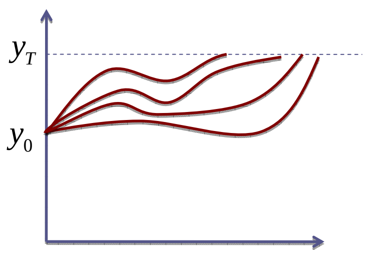
Control Óptimo

❖ **Condición de transversalidad**

$$[H]_{t=T} \Delta T - \lambda_T \Delta y_T = 0$$

Valor terminal fijo:

$$[H]_{t=T} = 0$$



Control Óptimo

❖ **Ejemplo 3: valor terminal fijo**

$$Max\ V[y_t] = \int_0^T (-t^2 - u_t^2)\partial t$$
$$s.a.:\quad \dot{y}_t = -u_t$$
$$y_0 = 4$$
$$y_T = 5$$

Condiciones de Primer Orden: $H = -t^2 - u_t^2 - \lambda_t u_t$

$$\frac{\partial H}{\partial u_t} = -2u_t - \lambda_t = 0$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial y_t} = 0$$

$$\dot{y}_t = \frac{\partial H}{\partial \lambda_t} = -u_t$$

$$\lambda_t^* = H_1$$

$$u_t^* = -\frac{H_1}{2}$$

$$y_t^* = \frac{H_1}{2}t + H_2$$

Control Óptimo

❖ **Ejemplo 3: valor terminal fijo**

$$Max\ V[y_t] = \int_0^T (t^2 - u_t^2)\partial t$$
$$s.a.:\quad \dot{y}_t = u_t$$
$$y_0 = 4$$
$$y_T = 5$$

Condiciones de Primer Orden: $y_t^* = \frac{H_1}{2}t + H_2$

Utilizando la condición inicial $y_t^* = \frac{H_1}{2}t + 4$

Cálculo del segundo parámetro $H(t, u_t, y_t, \lambda_t) = -t^2 - u_t^2 - \lambda_t u_t$

$$H(t, u_t, y_t, \lambda_t) = -t^2 - \left(-\frac{H_1}{2}\right)^2 - H_1\left(-\frac{H_1}{2}\right)$$

$$[H]_{t=T} = -T^2 + \left(\frac{H_1}{2}\right)^2 = 0$$

$$H_1 = 2T$$

Control Óptimo

❖ **Ejemplo 3: valor terminal fijo**

$$Max\ V[y_t] = \int_0^T (t^2 - u_t^2)\partial t$$

$$s.a.: \quad \dot{y}_t = u_t$$
$$y_0 = 4$$
$$y_T = 5$$

Condiciones de Primer Orden: $y_t^* = Tt + 4$

Cálculo la segunda condición $y_T^* = TT + 4 = 5$

$$T = 1$$

$$y_t^* = t + 4$$

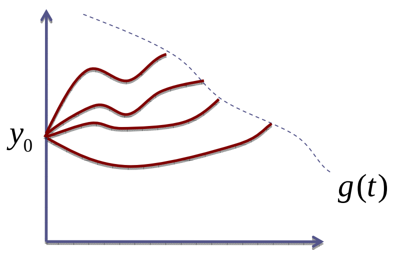
Control Óptimo

❖ **Condición de transversalidad**

$$[H]_{t=T}\,\Delta T - \lambda_T \Delta y_T = 0$$

Curva terminal fija:

$$[H]_{t=T} - \lambda_T g'(T) = 0$$



Control Óptimo

❖ **Ejemplo 4: curva terminal fija**

$$Max\ V[y_t] = \int_0^T (t^2 - u_t^2)\partial t$$

$$s.a.: \quad \dot{y}_t = u_t$$

$$y_0 = 4$$

$$y_T \ sobre\ la\ curva\ g(t) = 5.5 - \frac{t^2}{4}$$

Condiciones de Primer Orden:

$$H = t^2 - u_t^2 + \lambda_t u_t$$

$$\frac{\partial H}{\partial u_t} = -2u_t + \lambda_t = 0$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial y_t} = 0$$

$$\dot{y}_t = \frac{\partial H}{\partial \lambda_t} = u_t$$

$$\lambda_t^* = H_1$$

$$u_t^* = \frac{H_1}{2}$$

$$y_t^* = \frac{H_1}{2}t + H_2$$

Control Óptimo

❖ **Ejemplo 4: curva terminal fija**

$$Max\ V[y_t] = \int_0^T (t^2 - u_t^2)\partial t$$

$$s.a.: \quad \dot{y}_t = u_t$$

$$y_0 = 4$$

$$y_T \ sobre\ la\ curva\ g(t) = 5.5 - \frac{t^2}{2}$$

Condiciones de Primer Orden:

$$y_t^* = \frac{H_1}{2}t + H_2$$

Utilizando la condición inicial

$$y_t^* = \frac{H_1}{2}t + 4$$

Cálculo del segundo parámetro

$$[H]_{t=T} - \lambda_T g'(T) = 0$$

$$H(t, u_t, y_t, \lambda_t) = t^2 - \left(\frac{H_1}{2}\right)^2 + H_1\left(\frac{H_1}{2}\right)$$

$$\lambda_t g'(t) = H_1(-t)$$

$$[H]_{t=T} - \lambda_T g'(T) = T^2 + \left(\frac{H_1}{2}\right)^2 - H_1 T = 0$$

Control Óptimo

- **Ejemplo 4: curva terminal fija**

$$Max\ V[y_t] = \int_0^T (t^2 - u_t^2)\partial t$$

$$s.a.: \quad \dot{y}_t = u_t$$

$$y_0 = 4$$

$$y_T \text{ sobre la curva } g(t) = 5.5 - \frac{t^2}{2}$$

Condiciones de Primer Orden:

$$y_t^* = \frac{H_1}{2} t + 4$$

Cálculo del segundo parámetro

$$T^2 - 2\left(\frac{H_1}{2}\right)T + \left(\frac{H_1}{2}\right)^2 = 0$$

$$H_1 = 2T \qquad y_t^* = Tt + 4$$

$$y_T^* = T^2 + 4 \qquad g(T) = 5.5 - \frac{T^2}{2}$$

$$T = 1$$

$$y_t^* = t + 4$$

Control Óptimo

- **Condición de segundo orden**

$$H\left[f(t, y_t, u_t, \lambda_t) \right]_{(y^*, u^*)}$$

Método de menores principales

Método de raíces características

Control Óptimo

❖ Ejemplo 5: Minimización de costos

- Una fábrica recibe el pedido de B unidades (no hay plazo determinado)
- Costos cuadráticos de producción y costo de almacenaje proporcional a la producción acumulada.
- Definimos q la producción en período t, y la producción acumulada.

$$C_1(q_t) = aq_t^2 \qquad\qquad C_2(y_t) = by_t$$

- Teniendo en cuenta

$$q_t = \dot{y}_t$$

- Costo total

$$C(y_t) = a\dot{y}_t^2 + by_t$$

Control Óptimo

❖ Ejemplo 5: Minimización de costos (cálculo de variaciones)

$$Max\ V[y_t] = -\int_0^T (a\dot{y}_t^2 + by_t)\partial t$$

$$s.a.: \quad y_0 = 0$$

$$y_T = B$$

Condiciones de Primer Orden:

$$f_y = \frac{\partial f_{\dot{y}}}{\partial t}$$

$$f_y = -b$$

$$f_{\dot{y}} = -2a\dot{y}_t$$

$$\frac{\partial f_{\dot{y}}}{\partial t} = -2a\ddot{y}_t$$

$$2a\ddot{y}_t = b$$

$$y_t^* = \frac{b}{4a}t^2 + H_1 t + H_2$$

Control Óptimo

❖ **Ejemplo 5: Minimización de costos (control óptimo)**

$$Max\ V[y_t] = -\int_0^T (aq_t^2 + by_t)\partial t$$

$$s.a.: \quad \dot{y}_t = q_t$$

$$y_0 = 0$$

$$y_T = B$$

Condiciones de Primer Orden:

$$H = -aq_t^2 - by_t + \lambda_t q_t$$

$$\frac{\partial H}{\partial q_t} = -2aq_t + \lambda_t = 0$$

$$\lambda_t = 2aq_t \qquad \dot{\lambda}_t = 2a\dot{q}_t$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial y_t} = b$$

$$2a\dot{q}_t = b \qquad q_t = \frac{b}{2a}t + H_1$$

$$\dot{y}_t = \frac{\partial H}{\partial \lambda_t} = q_t$$

$$y_t^* = \frac{b}{4a}t^2 + H_1 t + H_2$$

Control Óptimo

❖ **Ejemplo 5: Minimización de costos**

$$Max\ V[y_t] = -\int_0^T (aq_t^2 + by_t)\partial t$$

$$s.a.: \quad \dot{y}_t = q_t$$

$$y_0 = 0$$

$$y_T = B$$

Condiciones de Primer Orden:

$$y_t^* = \frac{b}{4a}t^2 + H_1 t + H_2$$

Utilizando la condición inicial

$$y_t^* = \frac{b}{4a}t^2 + H_1 t$$

Cálculo del segundo parámetro

$$[H]_{t=T} = 0$$

$$H = -aq_t^2 - by_t + \lambda_t q_t = aq_t^2 - by_t = a\left(\frac{bt}{2a} + H_1\right)^2 - b\left(\frac{b}{4a}t^2 + H_1 t\right)$$

Control Óptimo

❖ **Ejemplo 5: Minimización de costos**

$$Max\ V[y_t] = -\int_0^T (aq_t^2 + by_t)\partial t$$

$$s.a.: \quad \dot{y}_t = q_t$$

$$y_0 = 0$$

$$y_T = B$$

Condiciones de Primer Orden:

$$y_t^* = \frac{b}{4a}t^2 + H_1 t$$

$$[H]_{t=T} = 0$$

$$H = aH_1^2$$

$$y_t^* = \frac{b}{4a}t^2$$

Control Óptimo

❖ **Horizonte temporal infinito**

$$MaxV = \int_0^\infty f(t, y_t, u_t)\partial t$$

$$s.a.: \quad \dot{y}_t = g(t, y_t, u_t)$$

$$y_0 \ \ dado$$

Condiciones de convergencia de la función objetivo:

1. Si la función V posee un valor finito hasta T, y luego toma el valor de cero, entonces la integral converge.

2. Si la función f es descontada a través del factor e^{-pt} (p>0) y tiene un valor menor o igual a M (finito), entonces la integral converge

Control Óptimo

❖ **Problema general**

$$MaxV = \int_0^\infty f(t, y_t, u_t)\partial t$$

$$s.a.: \quad \dot{y}_t = g(t, y_t, u_t)$$

$$y_0 \;\; dado$$

Condiciones de Primer Orden:

Condiciones de Segundo Orden:

Control Óptimo

❖ **Ejemplo 6:**

$$MaxV = \int_0^T Ln\ c_t \partial t$$

$$s.a.: \quad \dot{s}_t = rs_t - c_t$$

$$s(0) = s_0$$

$$s(T) = s_T$$

Condiciones de Primer Orden: $H(t, s_t, c_t) = Ln\ c_t + \lambda_t (rs_t - c_t)$

$$\underset{c_t}{Max}\, H \quad \rightarrow \quad \frac{1}{c_t} - \lambda_t = 0$$

$$\frac{\partial H}{\partial s_t} = -\dot{\lambda}_t \quad \rightarrow \quad r\lambda_t = -\dot{\lambda}_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{s}_t \quad \rightarrow \quad rs_t - c_t = \dot{s}_t$$

Control Óptimo

❖ **Ejemplo**

Resultado:

$$s_t = s_0 e^{rt} - \left(\frac{s_0 - s_T e^{-rT}}{T} \right) t e^{rt}$$

$$c_t = \left(\frac{s_0 - s_T e^{-rT}}{T} \right) e^{rt}$$

$$\lambda_t = \left(\frac{T}{s_0 - s_T e^{-rT}} \right) e^{-rt}$$

Condiciones de Segundo Orden:

$$Hes\left[H(t, y_t, u_t)\right]_{(y^*, u^*)} = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{c_t^2} \end{bmatrix}$$

Control Óptimo

❖ **Factor de descuento**

$$MaxV = \int_0^T e^{-\rho t} f(t, y_t, u_t) \partial t$$

$$s.a.: \quad \dot{y}_t = g(t, y_t, u_t)$$

$$y(0) = y_0$$

$$y(T) = y_T$$

Hamiltoniano ordinario: $H(t, y_t, u_t) = e^{-\rho t} f(t, y_t, u_t) + \lambda_t g(t, y_t, u_t)$

Hamiltoniano en tiempo presente: $H^* = e^{\rho t} H = f(t, y_t, u_t) + m_t g(t, y_t, u_t)$

$$\underset{u_t}{Max}\, H^*$$

$$\frac{\partial H^*}{\partial y_t} = -\dot{m}_t + \rho m_t$$

$$\frac{\partial H^*}{\partial m_t} = \dot{y}_t$$

$$m_T e^{\rho T} = 0$$

Control Óptimo

❖ **Ejemplo 6:**

$$MaxV = \int_0^T e^{-\rho t} Ln\ c_t \partial t$$

$$s.a.: \quad \dot{s}_t = rs_t - c_t$$

$$s(0) = s_0$$

$$s(T) = s_T$$

Condiciones de Primer Orden: $H(t, s_t, c_t) = e^{-\rho t} Ln\ c_t + \lambda_t (rs_t - c_t)$

$$\underset{c_t}{Max}\, H \quad \rightarrow \quad \frac{1}{c_t} e^{-\rho t} - \lambda_t = 0$$

$$\frac{\partial H}{\partial s_t} = -\dot{\lambda}_t \quad \rightarrow \quad r\lambda_t = -\dot{\lambda}_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{s}_t \quad \rightarrow \quad rs_t - c_t = \dot{s}_t$$

Control Óptimo

❖ **Ejemplo 6:**

$$MaxV = \int_0^T e^{-\rho t} Ln\ c_t \partial t$$

$$s.a.: \quad \dot{s}_t = rs_t - c_t$$

$$s(0) = s_0$$

$$s(T) = s_T$$

Condiciones de Primer Orden: $H^* = Ln\ c_t + m_t (rs_t - c_t)$

$$\underset{c_t}{Max}\, H^* \quad \rightarrow \quad \frac{1}{c_t} - m_t = 0$$

$$\frac{\partial H^*}{\partial s_t} = -\dot{m}_t + \rho m_t \quad \rightarrow \quad rm_t = -\dot{m}_t + \rho m_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{s}_t \quad \rightarrow \quad rs_t - c_t = \dot{s}_t$$

Control Óptimo

- **C.O. con Restricciones**

$$MaxV = \int_0^T f(t, y_t, u_t)\partial t$$

$$s.a.: \quad \dot{y}_t = g(t, y_t, u_t)$$

$$h(t, y_t, u_t) = 0$$

$$y(0) = y_0$$

$$y(T) = y_T$$

Hamiltoniano ordinario: $L = f(t, y_t, u_t) + \lambda_t g(t, y_t, u_t) + \gamma_t h(t, y_t, u_t)$

$$\underset{u_t}{Max}\, L \qquad \frac{\partial L}{\partial \lambda_t} = \dot{y}_t \qquad \frac{\partial L}{\partial \gamma_t} = 0$$

$$\frac{\partial L}{\partial y_t} = -\dot{\lambda}_t \qquad \lambda_T = 0$$

Control Óptimo

- **C.O. con múltiples variables de control**

$$MaxV = \int_0^T f(t, y_t, u_t, v_t)\partial t$$

$$s.a.: \quad \dot{y}_t = g(t, y_t, u_t, v_t)$$

$$y(0) = y_0$$

$$y(T) = y_T$$

Hamiltoniano ordinario: $H(t, y_t, u_t, v_t) = f(t, y_t, u_t, v_t) + \lambda_t g(t, y_t, u_t, v_t)$

$$\underset{u_t}{Max}\, H(t, y_t, u_t, v_t) \qquad \underset{v_t}{Max}\, H(t, y_t, u_t, v_t)$$

$$\frac{\partial H}{\partial y_t} = -\dot{\lambda}_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{y}_t$$

$$\lambda_T = 0$$

Control Óptimo

- **C.O. con múltiples variables de estado**

$$MaxV = \int_0^T f(t, y_t, z_t, u_t)\partial t$$

$$\begin{aligned} s.a.: \quad & \dot{y}_t = g(t, y_t, z_t, u_t) \\ & \dot{z}_t = h(t, y_t, z_t, u_t) \\ & y(0) = y_0 \\ & y(T) = y_T \end{aligned}$$

Hamiltoniano ordinario: $H(t, y_t, z_t, u_t) = f(t, y_t, z_t, u_t) + \lambda_t g(t, y_t, z_t, u_t) + \gamma_t h(t, y_t, z_t, u_t)$

$$\underset{u_t}{Max}\, H(t, y_t, z_t, u_t)$$

$$\frac{\partial H}{\partial y_t} = -\dot{\lambda}_t \qquad \frac{\partial H}{\partial z_t} = -\dot{\gamma}_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{y}_t \qquad \frac{\partial H}{\partial \gamma_t} = \dot{z}_t$$

$$\lambda_T = 0$$

Contenido

Programación Dinámica

❖ **Problema general**

$$Max\ V_t = \sum_{t=0}^{T} f_t(y_t,u_t) + Z(y_{T+1})$$

$$s.a.: \quad y_{t+1} \le g_t(y_t,u_t) \qquad \forall t = 0,1,\ldots,T$$

$$y(0) = y_0$$

$$y(T+1) = y_{T+1}$$

$$u_t, y_t \ge 0$$

Condiciones de Primer Orden: $L = \sum_{t=0}^{T} f_t(y_t,u_t) + Z(y_{T+1}) + \sum_{t=0}^{T} \lambda_t (g_t(y_t,u_t) - y_{t+1})$

$$\frac{\partial L}{\partial u_t} \le 0 \qquad u_t \ge 0 \qquad u_t \frac{\partial L}{\partial u_t} = 0$$

$$\frac{\partial L}{\partial y_{t+1}} \le 0 \qquad y_{t+1} \ge 0 \qquad y_{t+1} \frac{\partial L}{\partial y_{t+1}} = 0$$

$$\frac{\partial L}{\partial \lambda_t} \le 0 \qquad \lambda_t \ge 0 \qquad \lambda_t \frac{\partial L}{\partial \lambda_t} = 0$$

Programación Dinámica

❖ **Ecuación de Euler**

Asumiendo solución interna $L = \sum_{t=0}^{T} f_t(y_t,u_t) + Z(y_{T+1}) + \sum_{t=0}^{T} \lambda_t (g_t(y_t,u_t) - y_{t+1})$

$$\frac{\partial L}{\partial u_t} = 0 \rightarrow \frac{\partial L}{\partial u_t} = \frac{\partial f_t}{\partial u_t} + \lambda_t \frac{\partial g_t}{\partial u_t} = 0 \rightarrow \lambda_t = -\frac{\frac{\partial f_t}{\partial u_t}}{\frac{\partial g_t}{\partial u_t}} \qquad \lambda_{t+1} = -\frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}}$$

$$\frac{\partial L}{\partial y_{t+1}} = 0 \rightarrow \frac{\partial L}{\partial y_{t+1}} = \frac{\partial f_{t+1}}{\partial y_{t+1}} + \lambda_{t+1} \frac{\partial g_{t+1}}{\partial y_{t+1}} - \lambda_t = 0 \rightarrow \lambda_t = \frac{\partial f_{t+1}}{\partial y_{t+1}} + \lambda_{t+1} \frac{\partial g_{t+1}}{\partial y_{t+1}}$$

$$\frac{\partial L}{\partial \lambda_t} = 0 \rightarrow \frac{\partial L}{\partial \lambda_t} = g_t(y_t,u_t) - y_{t+1} = 0$$

Reemplazando en la primera condición

$$\frac{\partial f_t}{\partial u_t} + \frac{\partial g_t}{\partial u_t} \left[\frac{\partial f_{t+1}}{\partial y_{t+1}} - \frac{\partial g_{t+1}}{\partial y_{t+1}} \frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}} \right] = 0$$

Programación Dinámica

❖ **Problema general**

$$Max\ V_t = \sum_{t=0}^{T} f_t(y_t,u_t) + Z(y_{T+1})$$

$$s.a.: \quad y_{t+1} \leq g_t(y_t,u_t) \qquad \forall t = 0,1,\ldots,T$$

$$y(0) = y_0$$

$$y(T+1) = y_{T+1}$$

$$u_t, y_t \geq 0$$

Condiciones de Segundo Orden:

$$Hes\left[f_t(y_t,u_t)\right]_{(y^*,u^*)}$$

$$Hes\left[g_t(y_t,u_t)\right]_{(y^*,u^*)}$$

Programación Dinámica

❖ **Principio de optimalidad**

$$u_0^*,\ u_1^*,\ldots,\ u_t^*,\ u_{t+1}^*,\ldots,\ u_{T-1}^*,\ u_T^*$$

$$Max\ V_t = \sum_{t=0}^{T} f_t(y_t,u_t) + Z(y_{T+1})$$

$$s.a.: \quad y_{t+1} \leq g_t(y_t,u_t) \qquad \forall t = 0,1,\ldots,T$$

$$y_0 \ \ dado$$

$$y_{T+1} \ \ dado$$

$$Max\ V_t = \sum_{\tau=t}^{T} f_\tau(y_\tau,u_\tau) + Z(y_{T+1})$$

$$s.a.: \quad y_{\tau+1} \leq g_\tau(y_\tau,u_\tau) \qquad \forall \tau = t,t+1,\ldots,T$$

$$y_t \ \ dado$$

$$y_{T+1} \ \ dado$$

Programación Dinámica

❖ Ecuación de Bellman

$$Max\ V_t = \sum_{\tau=t}^{T} f_\tau(y_\tau, u_\tau) + Z(y_{T+1})$$

$$s.a.: \quad y_{\tau+1} \le g_\tau(y_\tau, u_\tau) \qquad \forall \tau = t, t+1, ..., T$$

$$y_t \ dado$$

$$y_{T+1} \ dado$$

$$V_t(y_t) = \underset{u_t}{Max}\left[f_t(y_t, u_t) + V_{t+1}(y_{t+1})\right]$$

$$s.a.: \quad y_{t+1} \le g_t(y_t, u_t) \qquad \forall t = 0,1,...,T$$

$$y_0 \ dado$$

$$y_{T+1} \ dado$$

❖ Ecuación de Benveniste y Scheinkman

De Bellman

$$\frac{\partial f_t}{\partial u_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial u_t} = 0$$

$$y_{t+1} = g_t(y_t, u_t)$$

Función de política

$$u_t = h(y_t)$$

Teorema de la envolvente

$$V_t(y_t) = f_t(y_t, h(y_t)) + V_{t+1}(g_t(y_t, h(y_t)))$$

$$\frac{\partial V_t}{\partial y_t} = \frac{\partial f_t}{\partial y_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial y_t}$$

Programación Dinámica

❖ Ecuación de Euler

De la CPO de Bellman

$$\frac{\partial f_t}{\partial u_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial u_t} = 0 \longrightarrow \frac{\partial V_{t+1}}{\partial y_{t+1}} = -\frac{\dfrac{\partial f_t}{\partial u_t}}{\dfrac{\partial g_t}{\partial u_t}} \longrightarrow \frac{\partial V_{t+2}}{\partial y_{t+2}} = -\frac{\dfrac{\partial f_{t+1}}{\partial u_{t+1}}}{\dfrac{\partial g_{t+1}}{\partial u_{t+1}}}$$

De Benveniste y Scheinkman

$$\frac{\partial V_t}{\partial y_t} = \frac{\partial f_t}{\partial y_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial y_t} \longrightarrow \frac{\partial V_{t+1}}{\partial y_{t+1}} = \frac{\partial f_{t+1}}{\partial y_{t+1}} + \frac{\partial V_{t+2}}{\partial y_{t+2}} \frac{\partial g_{t+1}}{\partial y_{t+1}}$$

En la CPO

$$\frac{\partial f_t}{\partial u_t} + \frac{\partial g_t}{\partial u_t}\left[\frac{\partial f_{t+1}}{\partial y_{t+1}} - \frac{\partial g_{t+1}}{\partial y_{t+1}} \frac{\dfrac{\partial f_{t+1}}{\partial u_{t+1}}}{\dfrac{\partial g_{t+1}}{\partial u_{t+1}}}\right] = 0$$

Programación Dinámica

❖ **Ejemplo:**

$$Max\ V_t = \sum_{t=0}^{T} \beta^t Ln\ c_t$$

$$s.a.: \quad s_{t+1} \le (1+r)(s_t - c_t) \qquad \forall t = 0,1,\ldots,T$$

$$s(0) = s_0$$

$$s_{T+1} \ libre$$

$$c_t, s_t \ge 0$$

Ecuación de Euler: $\dfrac{\partial f_t}{\partial u_t} + \dfrac{\partial g_t}{\partial u_t}\left[\dfrac{\partial f_{t+1}}{\partial y_{t+1}} - \dfrac{\partial g_{t+1}}{\partial y_{t+1}}\dfrac{\dfrac{\partial f_{t+1}}{\partial u_{t+1}}}{\dfrac{\partial g_{t+1}}{\partial u_{t+1}}}\right] = 0$

$$\frac{\beta^t}{c_t} - (1+r)\left[0 + (1+r)\frac{\dfrac{\beta^{t+1}}{c_{t+1}}}{(1+r)}\right] = 0$$

$$\boxed{c_{t+1} = \beta(1+r)c_t}$$

Programación Dinámica

❖ **Ejemplo:** $s_{t+1} = (1+r)(s_t - c_t)$ ...Restricción

$c_{t+1} = \beta(1+r)c_t$ ...Ecuación de Euler

Función de Política: $c_t = h(s_t)$

En T: $s_{T+1} = (1+r)(s_T - c_T) = 0$ $\qquad \boxed{c_T = s_T}$

En T-1: $s_T = (1+r)(s_{T-1} - c_{T-1})$

$c_T = \beta(1+r)c_{T-1}$

$c_T = (1+r)(s_{T-1} - c_{T-1})$

$$\boxed{c_{T-1} = \frac{s_{T-1}}{1+\beta}}$$

En T-2: $s_{T-1} = (1+r)(s_{T-2} - c_{T-2})$

$c_{T-1} = \beta(1+r)c_{T-2}$

$(1+\beta)c_T = (1+r)(s_{T-2} - c_{T-2})$

$$\boxed{c_{T-2} = \frac{s_{T-2}}{1+\beta+\beta^2}}$$

En T-i: $\boxed{c_{T-i} = \dfrac{s_{T-i}}{1+\beta+\beta^2+\ldots+\beta^i}}$

Programación Dinámica

❖ **Programación dinámica con incertidumbre**

$$
\begin{aligned}
&Max\ V_t = E_0 \sum_{t=0}^{T} f_t(y_t, u_t) \\
&s.a.: \quad y_{t+1} \le g_t(y_t, u_t, \varepsilon_{t+1}) \qquad \forall t = 0,1,\dots,T \\
&\qquad y(0) = y_0 \\
&\qquad y(T+1) = y_{T+1} \\
&\qquad u_t, y_t \ge 0
\end{aligned}
$$

❖ **Ecuación de Bellman**

$$
\begin{aligned}
&V_t(y_t) = \underset{u_t}{Max}\left[f_t(y_t, u_t) + E_t V_{t+1}(y_{t+1}) \right] \\
&s.a.: \quad y_{t+1} \le g_t(y_t, u_t, \varepsilon_{t+1}) \qquad \forall t = 0,1,\dots,T \\
&\qquad y_0 \ \ dado \\
&\qquad y_{T+1} \ \ dado
\end{aligned}
$$

Programación Dinámica

❖ **Ecuación de Bellman**

$$
\begin{aligned}
&V_t(y_t) = \underset{u_t}{Max}\left[f_t(y_t, u_t) + E_t V_{t+1}(y_{t+1}) \right] \\
&s.a.: \quad y_{t+1} \le g_t(y_t, u_t, \varepsilon_{t+1}) \qquad \forall t = 0,1,\dots,T \\
&\qquad y_0 \ \ dado \\
&\qquad y_{T+1} \ \ dado
\end{aligned}
$$

❖ **Ecuación de Benveniste y Scheinkman**

De Bellman

$$
\frac{\partial f_t}{\partial u_t} + E_t\left[\frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial u_t} \right] = 0
$$

$$
y_{t+1} = g_t(y_t, u_t, \varepsilon_{t+1})
$$

Plan contingente

$$
u_t = h(y_t)
$$

Teorema de la envolvente

$$
W_t(y_t) = f_t(y_t, h(y_t)) + E_t\left[W_{t+1}(g_t(y_t, h(y_t), \varepsilon_{t+1})) \right]
$$

$$
\frac{\partial V_t}{\partial y_t} = \frac{\partial f_t}{\partial y_t} + E_t\left[\frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial y_t} \right]
$$

Programación Dinámica

❖ **Ecuación de Euler: Si** $\boxed{y_{t+1} \leq g_t(u_t)}$ $\quad \forall t = 0,1,\ldots,T$

De la CPO de Bellman

$$\frac{\partial f_t}{\partial u_t} + E_t\left[\frac{\partial V_{t+1}}{\partial y_{t+1}}\frac{\partial g_t}{\partial u_t}\right] = 0$$

De Benveniste y Scheinkman

$$\frac{\partial V_t}{\partial y_t} = \frac{\partial f_t}{\partial y_t} \quad \longrightarrow \quad \frac{\partial V_{t+1}}{\partial y_{t+1}} = \frac{\partial f_{t+1}}{\partial y_{t+1}}$$

En la CPO

$$\boxed{\frac{\partial f_t}{\partial u_t} + E_t\left[\frac{\partial f_{t+1}}{\partial y_{t+1}}\frac{\partial g_t}{\partial u_t}\right] = 0}$$

UNIVERSIDAD SAN IGNACIO DE LOYOLA

Economía Matemática Dinámica

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