



Repaso Matemático II

Ronald Cuela

Contenido

- 1 Maximización Estática
- 2 Cálculo de Variaciones
- 3 Control Óptimo
- 4 Programación Dinámica



Maximización Estática

❖ Problema general sin restricciones

$$\underset{x_i}{\text{Max}} f(x_1, x_2, \dots, x_n) \rightarrow i = 1, 2, \dots, n$$

Condiciones de Primer Orden:

$$\frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_i} = 0 \rightarrow i = 1, 2, \dots, n$$

Condiciones de Segundo Orden:

$$H[f(x_1, x_2, \dots, x_n)]_{x=x^*}$$



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Maximización Estática

❖ Problema general con restricciones de igualdad

$$\underset{x_i}{\text{Max}} f(x_1, x_2, \dots, x_n) \rightarrow i = 1, 2, \dots, n$$

$$\text{s.a.: } g_j(x_1, x_2, \dots, x_n) = 0 \rightarrow j = 1, 2, \dots, m$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^m \lambda_j g_j(x_i)$

$$\frac{\partial L}{\partial x_i} = \frac{\partial f(x_i)}{\partial x_i} + \sum_{j=1}^m \left[\lambda_j \frac{\partial g_j(x_i)}{\partial x_i} \right] = 0 \rightarrow i = 1, 2, \dots, n$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(x_1, x_2, \dots, x_n) = 0 \rightarrow j = 1, 2, \dots, m$$

Condiciones de Segundo Orden:

$$H[f(x_1, x_2, \dots, x_n)]_{x=x^*}$$

$$H[g(x_1, x_2, \dots, x_m)]_{x=x^*}$$



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Maximización Estática

❖ Problema general con restricciones de desigualdad

$$\underset{x_i}{\text{Max}} f(x_1, x_2, \dots, x_n) \rightarrow i = 1, 2, \dots, n$$

$$\text{s.a.: } g_j(x_1, x_2, \dots, x_n) \geq 0 \rightarrow j = 1, 2, \dots, m$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^m \lambda_j g_j(x_i)$

$$\frac{\partial L}{\partial x_i} = 0 \rightarrow i = 1, 2, \dots, n$$

$$\lambda_j \geq 0 \quad \frac{\partial L}{\partial \lambda_j} \geq 0 \quad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0 \rightarrow j = 1, 2, \dots, m$$

Condiciones de Segundo Orden:

$$H[f(x_1, x_2, \dots, x_n)]_{x=x^*}$$

$$H[g(x_1, x_2, \dots, x_m)]_{x=x^*}$$



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Maximización Estática

❖ Problema general con restricciones de desigualdad y no negatividad

$$\underset{x_i}{\text{Max}} f(x_1, x_2, \dots, x_n) \rightarrow i = 1, 2, \dots, n$$

$$\text{s.a.: } g_j(x_1, x_2, \dots, x_n) \geq 0 \rightarrow j = 1, 2, \dots, m$$

$$x_i \geq 0 \rightarrow i = 1, 2, \dots, n$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^m \lambda_j g_j(x_i)$

$$x_i \geq 0 \quad \frac{\partial L}{\partial x_i} \leq 0 \quad x_i \frac{\partial L}{\partial x_i} = 0 \rightarrow i = 1, 2, \dots, n$$

$$\lambda_j \geq 0 \quad \frac{\partial L}{\partial \lambda_j} \geq 0 \quad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0 \rightarrow j = 1, 2, \dots, m$$

Condiciones de Segundo Orden:

$$H[f(x_1, x_2, \dots, x_n)]_{x=x^*}$$

$$H[g(x_1, x_2, \dots, x_m)]_{x=x^*}$$



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Cálculo de variaciones

❖ Problema general

$$\text{Max}V = \int_0^T f(t, y_t, \dot{y}_t) \partial t$$

$$\text{s.a.: } y(0) = y_0$$

$$y(T) = y_T$$

Condiciones de Primer Orden:

$$f_y = \frac{\partial f}{\partial \dot{y}}$$

Condiciones de Segundo Orden:

$$H[f(t, y_t, \dot{y}_t)]_{(y^*, \dot{y}^*)}$$



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Cálculo de variaciones

❖ Condición de transversalidad

$$\left[f_{\dot{y}} \right]_{t=T} \Delta y_T + \left[f - \dot{y} f_{\dot{y}} \right]_{t=T} \Delta T = 0$$

Horizonte temporal fijo:

$$\left[f_{\dot{y}} \right]_{t=T} = 0$$

Valor terminal fijo:

$$\left[f - \dot{y} f_{\dot{y}} \right]_{t=T} = 0$$

Curva terminal:

$$\left[f_{\dot{y}} \right]_{t=T} g'(T) + \left[f - \dot{y} f_{\dot{y}} \right]_{t=T} = 0$$



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Control Óptimo

❖ Problema general

$$\begin{aligned} \text{Max} V &= \int_0^T f(t, y_t, u_t) \partial t \\ \text{s.a.: } \dot{y}_t &= g(t, y_t, u_t) \\ y(0) &= y_0 \\ y(T) &= y_T \end{aligned}$$

Condiciones de Primer Orden: $H(t, y_t, u_t) = f(t, y_t, u_t) + \lambda_t g(t, y_t, u_t)$

$$\text{Max}_{u_t} H(t, y_t, u_t)$$

$$\frac{\partial H}{\partial y_t} = -\dot{\lambda}_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{y}_t$$

Condiciones de Segundo Orden: $\text{Hes}[H(t, y_t, u_t)]_{(y^*, u^*)}$



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Control Óptimo

❖ Condición de transversalidad

$$[H]_{t=T} \Delta T - \lambda_T \Delta y_T = 0$$

Horizonte temporal fijo:

$$\lambda_T = 0$$

Valor terminal fijo:

$$[H]_{t=T} = 0$$

Curva terminal:

$$[H]_{t=T} - \lambda_T g'(T) = 0$$



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Control Óptimo

❖ Ejemplo

$$\begin{aligned} \text{Max} V &= \int_0^T L n c_t \delta t \\ \text{s.a.: } \dot{s}_t &= r s_t - c_t \\ s(0) &= s_0 \\ s(T) &= s_T \end{aligned}$$

Condiciones de Primer Orden: $H(t, s_t, c_t) = L n c_t + \lambda_t (r s_t - c_t)$

$$\text{Max}_{c_t} H \quad \rightarrow \quad \frac{1}{c_t} - \lambda_t = 0$$

$$\frac{\partial H}{\partial s_t} = -\dot{\lambda}_t \quad \rightarrow \quad r \lambda_t = -\dot{\lambda}_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{s}_t \quad \rightarrow \quad r s_t - c_t = \dot{s}_t$$



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Control Óptimo

❖ Ejemplo

Resultado:

$$s_t = s_0 e^{rt} - \left(\frac{s_0 - s_T e^{-rT}}{T} \right) t e^{rt}$$

$$c_t = \left(\frac{s_0 - s_T e^{-rT}}{T} \right) e^{rt}$$

$$\lambda_t = \left(\frac{T}{s_0 - s_T e^{-rT}} \right) e^{-rt}$$

Condiciones de Segundo Orden:

$$\text{Hes}[H(t, y_t, u_t)]_{(y^*, u^*)} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{c_t^2} \end{bmatrix}$$



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Programación Dinámica

❖ Problema general

$$\text{Max } V_t = \sum_{t=0}^T f_t(y_t, u_t) + Z(y_{T+1})$$

$$\text{s.a.: } y_{t+1} \leq g_t(y_t, u_t) \quad \forall t = 0, 1, \dots, T$$

$$y(0) = y_0$$

$$y(T+1) = y_{T+1}$$

$$u_t, y_t \geq 0$$

Condiciones de Primer Orden: $L = \sum_{t=0}^T f_t(y_t, u_t) + Z(y_{T+1}) + \sum_{t=0}^T \lambda_t (g_t(y_t, u_t) - y_{t+1})$

$$\frac{\partial L}{\partial u_t} \leq 0 \quad u_t \geq 0 \quad u_t \frac{\partial L}{\partial u_t} = 0$$

$$\frac{\partial L}{\partial y_{t+1}} \leq 0 \quad y_{t+1} \geq 0 \quad y_{t+1} \frac{\partial L}{\partial y_{t+1}} = 0$$

$$\frac{\partial L}{\partial \lambda_t} \leq 0 \quad \lambda_t \geq 0 \quad \lambda_t \frac{\partial L}{\partial \lambda_t} = 0$$



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Programación Dinámica

❖ Ecuación de Euler

Asumiendo solución interna $L = \sum_{t=0}^T f_t(y_t, u_t) + Z(y_{T+1}) + \sum_{t=0}^T \lambda_t (g_t(y_t, u_t) - y_{t+1})$

$$\frac{\partial L}{\partial u_t} = 0 \rightarrow \frac{\partial L}{\partial u_t} = \frac{\partial f_t}{\partial u_t} + \lambda_t \frac{\partial g_t}{\partial u_t} = 0$$

$$\rightarrow \lambda_t = - \frac{\frac{\partial f_t}{\partial u_t}}{\frac{\partial g_t}{\partial u_t}} \quad \lambda_{t+1} = - \frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}}$$

$$\frac{\partial L}{\partial y_{t+1}} = 0 \rightarrow \frac{\partial L}{\partial y_{t+1}} = \frac{\partial f_{t+1}}{\partial y_{t+1}} + \lambda_{t+1} \frac{\partial g_{t+1}}{\partial y_{t+1}} - \lambda_t = 0 \rightarrow$$

$$\lambda_t = \frac{\partial f_{t+1}}{\partial y_{t+1}} + \lambda_{t+1} \frac{\partial g_{t+1}}{\partial y_{t+1}}$$

$$\frac{\partial L}{\partial \lambda_t} = 0 \rightarrow \frac{\partial L}{\partial \lambda_t} = g_t(y_t, u_t) - y_{t+1} = 0$$

Reemplazando en la primera condición

$$\frac{\partial f_t}{\partial u_t} + \frac{\partial g_t}{\partial u_t} \left[\frac{\partial f_{t+1}}{\partial y_{t+1}} - \frac{\partial g_{t+1}}{\partial y_{t+1}} \frac{\partial u_{t+1}}{\partial g_{t+1}} \right] = 0$$



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Programación Dinámica

❖ Problema general

$$\text{Max } V_t = \sum_{t=0}^T f_t(y_t, u_t) + Z(y_{T+1})$$

$$\text{s.a.: } y_{t+1} \leq g_t(y_t, u_t) \quad \forall t = 0, 1, \dots, T$$

$$y(0) = y_0$$

$$y(T+1) = y_{T+1}$$

$$u_t, y_t \geq 0$$

Condiciones de Segundo Orden:

$$\text{Hes}[f_t(y_t, u_t)]_{(y^*, u^*)}$$

$$\text{Hes}[g_t(y_t, u_t)]_{(y^*, u^*)}$$



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❖ Principio de optimalidad

$$\underline{u_0^*, u_1^*, \dots, u_t^*, u_{t+1}^*, \dots, u_{T-1}^*, u_T^*}$$

$Max V_t = \sum_{\tau=t}^T f_\tau(y_\tau, u_\tau) + Z(y_{T+1})$
 s.a.: $y_{t+1} \leq g_t(y_t, u_t) \quad \forall t=0,1,\dots,T$
 y_0 *dado*
 y_{T+1} *dado*

$Max V_t = \sum_{\tau=t}^T f_\tau(y_\tau, u_\tau) + Z(y_{T+1})$
 s.a.: $y_{\tau+1} \leq g_\tau(y_\tau, u_\tau) \quad \forall \tau=t, t+1, \dots, T$
 y_t *dado*
 y_{T+1} *dado*

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❖ Ecuación de Bellman

$Max V_t = \sum_{\tau=t}^T f_\tau(y_\tau, u_\tau) + Z(y_{T+1})$
 s.a.: $y_{\tau+1} \leq g_\tau(y_\tau, u_\tau) \quad \forall \tau=t, t+1, \dots, T$
 y_t *dado*
 y_{T+1} *dado*

$V_t(y_t) = Max_{u_t} [f_t(y_t, u_t) + V_{t+1}(y_{t+1})]$
 s.a.: $y_{t+1} \leq g_t(y_t, u_t) \quad \forall t=0,1,\dots,T$
 y_0 *dado*
 y_{T+1} *dado*

❖ Ecuación de Benveniste y Scheinkman

De Bellman

$\frac{\partial f_t}{\partial u_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial u_t} = 0$
 $y_{t+1} = g_t(y_t, u_t)$

Función de política

$u_t = h(y_t)$

Teorema de la envolvente

$V_t(y_t) = f_t(y_t, h(y_t)) + V_{t+1}(g_t(y_t, h(y_t)))$

$\frac{\partial V_t}{\partial y_t} = \frac{\partial f_t}{\partial y_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial y_t}$

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❖ Ecuación de Euler

De la CPO de Bellman

$$\frac{\partial f_t}{\partial u_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial u_t} = 0 \quad \rightarrow \quad \frac{\partial V_{t+1}}{\partial y_{t+1}} = - \frac{\frac{\partial f_t}{\partial u_t}}{\frac{\partial g_t}{\partial u_t}} \quad \rightarrow \quad \frac{\partial V_{t+2}}{\partial y_{t+2}} = - \frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}}$$

De Benveniste y Scheinkman

$$\frac{\partial V_t}{\partial y_t} = \frac{\partial f_t}{\partial y_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}} \frac{\partial g_t}{\partial y_t} \quad \rightarrow \quad \frac{\partial V_{t+1}}{\partial y_{t+1}} = \frac{\partial f_{t+1}}{\partial y_{t+1}} + \frac{\partial V_{t+2}}{\partial y_{t+2}} \frac{\partial g_{t+1}}{\partial y_{t+1}}$$

En la CPO

$$\frac{\partial f_t}{\partial u_t} + \frac{\partial g_t}{\partial u_t} \left[\frac{\partial f_{t+1}}{\partial y_{t+1}} - \frac{\partial g_{t+1}}{\partial y_{t+1}} \frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}} \right] = 0$$



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❖ Ejemplo:

$$\text{Max } V_t = \sum_{t=0}^T \beta^t \text{Ln } c_t$$

$$\text{s.a.: } s_{t+1} \leq (1+r)(s_t - c_t) \quad \forall t = 0, 1, \dots, T$$

$$s(0) = s_0$$

$$s_{T+1} \text{ libre}$$

$$c_t, s_t \geq 0$$

Ecuación de Euler:
$$\frac{\partial f_t}{\partial u_t} + \frac{\partial g_t}{\partial u_t} \left[\frac{\partial f_{t+1}}{\partial y_{t+1}} - \frac{\partial g_{t+1}}{\partial y_{t+1}} \frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}} \right] = 0$$

$$\frac{\beta^t}{c_t} - (1+r) \left[0 + (1+r) \frac{\beta^{t+1}}{(1+r)} \right] = 0$$

$$c_{t+1} = \beta(1+r)c_t$$



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❖ **Ejemplo:** $s_{t+1} = (1+r)(s_t - c_t)$...Restricción
 $c_{t+1} = \beta(1+r)c_t$...Ecuación de Euler

Función de Política: $c_t = h(s_t)$

En T: $s_{T+1} = (1+r)(s_T - c_T) = 0$

$$c_T = s_T$$

En T-1: $s_T = (1+r)(s_{T-1} - c_{T-1})$

$$c_T = (1+r)(s_{T-1} - c_{T-1})$$

$$c_T = \beta(1+r)c_{T-1}$$

$$c_{T-1} = \frac{s_{T-1}}{1+\beta}$$

En T-2: $s_{T-1} = (1+r)(s_{T-2} - c_{T-2})$

$$(1+\beta)c_T = (1+r)(s_{T-2} - c_{T-2})$$

$$c_{T-1} = \beta(1+r)c_{T-2}$$

$$c_{T-2} = \frac{s_{T-2}}{1+\beta+\beta^2}$$

En T-i:

$$c_{T-i} = \frac{s_{T-i}}{1+\beta+\beta^2+\dots+\beta^i}$$



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